INITIAL-VALUE PROBLEM FOR THE MOTION IN AN UNDULATING SEA OF A BODY WITH FIXED EQUILIBRIUM POSITION*)

by

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1. *Introduction*

We shall consider a body moving on or beneath the free surface of a heavy inviscid fluid. The initial position and velocity of both fluid and body are assumed to be known, and we further allow the possible presence of 'known waves' which may diffract upon the body and also cause it to move if its motion is not constrained. The motion of the fluid at the initial instant will be assumed to be irrotational. Hence it remains so. The body is supposed to have zero average velocity in both translation and rotation and its motions are assumed small enough so that the boundary conditions to be satisfied on the body can be linearized and satisfied at its equilibrium position. Concomitant with this, we assume that the boundary conditions on the free surface can also be linearized. We shall be concerned with various aspects of the motion after the initial instant.

The method of analysis which is used was introduced by Volterra (1934), has been carried further by Finkelstein (1957), and is expounded in both Stoker's *Water waves* (1957, pp. 187-196) and in Wehausen and Laitone's *Gravity waves* (1960, pp. 603-607). The novelty here consists in certain decompositions of the velocity potential which allow one to derive Cummin's (1962) results for the initial-value problem with no waves present by what seems to the author to be a more direct approach and at the same time to find analogues for unsteady motion of the Haskind relations (1957; see also Newman, 1962) between the force and moment acting on the body associated with diffracted waves and with forced waves.

In the last section the various forces and moments acting on the body are put together in the equations of motion for the (small) translational and rotational motions of the body. These take the form of six coupled integro-differential equations. Certain coefficients and kernels occurring in these equations require prior solution of integral equations in which the shape of the body but not its motion is involved. This is one of the advantages of linearization. The solution of the equations themselves is not considered here. We mention, however, that Ursell (1964) has treated the initial-value problem for a half-immersed circular cylinder in still water, but by a quite different method from that used here. Earlier, Sretenskii (1937) had derived an integro-differential equation for a special case of the present problem and solved it numerically for a particular body.

The integral equations mentioned above are discussed in Appendix III, where it is shown that they can be reduced to Fredholm integral equations. No attempt is made to establish the existence of a solution. Uniqueness of solution follows easily from an extension of Volterra's original treatment of this problem.

Some of the results, for example, the Cummins decomposition, can be extended to allow the body to have a constant translational velocity. This has recently been done by Wen-Chin Lin [Ph.D. dissertation, University of California, Berkeley, 1966].

2. *Mathematical formulation*

Let Oxyz be an inertial right-handed coordinate system with Oy directed oppositely to gravity and with Oxz lying in the plane of the undisturbed free

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surface. Let $\hat{O}\hat{x}\hat{y}\hat{z}$ be a system fixed in the body and coinciding with Oxyz when the body is at rest in its equilibrium position. Then, if excursions of the body from its equilibrium position are 'small', and if one discards terms higher than first order in the excursions, the two systems are related by the following equation:

$$\begin{aligned}(x, y, z) &= (\hat{x}, \hat{y}, \hat{z}) + (x_1(t), y_1(t), z_1(t)) + (\alpha(t), \beta(t), \gamma(t)) \times (\hat{x}, \hat{y}, \hat{z}) \\ &= (\hat{x}, \hat{y}, \hat{z}) + (x_1, y_1, z_1) + (\alpha, \beta, \gamma) \times (x, y, z) \end{aligned} \tag{1a}$$

which we may also write in the following form:

$$\underline{r} = \hat{\underline{r}} + \sum_{k=1}^{3} \left[\alpha_k(t)\, \underline{e}_k + \alpha_{k+3}(t)\, \underline{e}_k \times \underline{r}\right], \tag{1b}$$

where $\underline{e}_1, \underline{e}_2, \underline{e}_3$, are the unit vectors in the directions Ox, Oy, Oz, respectively, and

$$\alpha_1 = x_1, \quad \alpha_2 = y_1, \quad \alpha_3 = z_1, \quad \alpha_4 = \alpha, \quad \alpha_5 = \beta, \quad \alpha_6 = \gamma.$$

Here (x_1, y_1, z_1) describes the translational displacements and (α, β, γ) the angular ones. Henceforth we shall not need the coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$.

The motion of the fluid may be described by means of a velocity potential $\phi(x, y, z, t)$. The linearized boundary conditions which it must satisfy are the following:

$$\phi_{tt}(x, o, z, t) + g\phi_y = 0, \tag{2a}$$

$$\begin{aligned}\phi_n\big|_S = v_n(x, y, z, t) &= \sum_{k=1}^{3} \left[\dot{\alpha}_k(t)\underline{n} \cdot \underline{e}_k + \dot{\alpha}_{k+3}(t)\,(\underline{r} \times \underline{n}) \cdot \underline{e}_k\right] \\ &= \sum_{k=1}^{6} \dot{\alpha}_k(t) n_k, \end{aligned} \tag{2b}$$

$$\phi_n\big|_B = 0, \tag{2c}$$

where S is the wetted surface of the body in its equilibrium position and B the bottom. We shall always take the normal vector $\underline{n}$ to be pointing out of the fluid. The components n_4, n_5, n_6 are defined by

$$n_k = (\underline{r} \times \underline{n}) \cdot \underline{e}_{k-3}, \quad k = 4, 5, 6.$$

If the fluid is infinitely deep, the last condition is replaced by

$$\lim_{y \to -\infty} \phi_y = 0. \tag{2d}$$

In addition, we require that ϕ, ϕ_t, grad ϕ be bounded in the region occupied by fluid.

We shall further suppose that $\phi(x, y, z, o)$ and $\phi_t(x, o, z, o)$ are known. $Y(x, z, t)$, the free surface elevation, is given in linearized theory by

$$Y(x, z, t) = -\frac{1}{g}\, \phi_t(x, o, z, t), \tag{3}$$

and $Y_t(x, z, t)$ by

$$Y_t(x, z, t) = \phi_y(x, o, z, t).$$

3. *A preliminary decomposition*

We shall attempt to separate the velocity potential $\phi(P, t)$ into two parts, a 'forced-wave' potential ϕ_F, representing the waves caused by the motion of the floating body, and a 'free-wave' potential ϕ_W representing the wave motion which would take place if the body were not moving. In order to describe ϕ_W we must know an 'incoming-wave' potential $\phi_I(P, t)$. This might be, typically, plane sinusoidal travelling, or standing, waves of given length, but could also be a more complex sea representable by a Fourier integral or generalized Fourier series in plane-wave potentials of variable direction and wave length. The important point is that $\phi_I(P, t)$ should be known for $t \geqq 0$. Associated with ϕ_I will be another function $\phi_D(P, t)$, the 'diffracted-wave' potential, which must satisfy the boundary condition

$$\phi_{Dn}(P, t)|_S = -\phi_{In}(P, t)|_S \,. \tag{4}$$

We now define ϕ_W by

$$\phi_W = \phi_I + \phi_D. \tag{5}$$

Evidently

$$\phi_{Wn}(P, t)|_S = 0, \quad t \geqq 0. \tag{6}$$

Since we wish to have

$$\phi = \phi_F + \phi_W , \tag{7}$$

ϕ_F must satisfy

$$\phi_{Fn}(P, t)|_S = v_n , \quad t \geqq 0. \tag{8}$$

All functions ϕ_I, ϕ_D and ϕ_F must, of course, satisfy condition (2a) and (2c) and be bounded.

We shall suppose that at time t=0 we know $\phi_F(x, y, z, o)$, $\phi_{Ft}(x, o, z, o)$, $\phi_D(x, y, z, o)$, and $\phi_{Dt}(x, o, z, o)$. Hence, we also know $Y_F(x, z, o)$, $Y_{Ft}(x, z, o)$, $Y_W(x, z, o)$ and $Y_{Wt}(x, z, o)$; where these are determined according to (3) from the corresponding velocity potential.

4. *Volterra's method*

Let $G(x, y, z; \xi, \eta, \zeta; t) = G(P; Q; t)$ be a Green function defined in $y \leq 0$, $\eta \leq 0$ which, in addition to being of the form

$$G = r^{-1} + H(P;Q;t), \quad r = \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{\frac{1}{2}} , \tag{9}$$

where H is harmonic in the region of definition, satisfies the following conditions:

$$\begin{aligned}
&G_{tt}(P; \xi, o, \zeta; t) + gG_\eta = 0,\\
&G_\nu = 0 \quad \text{for} \quad Q \in B,\\
&G(P; \xi, o, \zeta; o) = 0,\\
&G(P; Q; -t) = G(P; Q; t),\\
&G = O(R^{-2}), \; G_R = O(R^{-3}) \quad \text{if} \quad R \to \infty ,
\end{aligned} \tag{10}$$

where $R=\left[(x-\xi)^2+(z-\zeta)^2\right]^{\frac{1}{2}}$, and

$$\frac{\partial}{\partial\nu}=n_1(Q)\,\frac{\partial}{\partial\xi}+n_2(Q)\,\frac{\partial}{\partial\eta}+n_3(Q)\,\frac{\partial}{\partial\zeta}$$

Such functions can be constructed for infinitely deep fluid and for the case of a horizontal bottom at $y=-h$ when the fluid is not bounded in any horizontal direction [see, e.g., Wehausen and Laitone (1960), p. 604]. The method of images allows easy extension to certain cases with vertical walls. In all cases,

$$G(P;Q;t)=G(Q;P;t). \tag{11}$$

Volterra's method for the problem at hand starts out by applying Green's Theorem to the functions $\phi_t(x,y,z,\tau)$ and $G(P;Q;t-\tau)$ in the region bounded by the free surface F, the body surface S, the bottom B, and a large vertical cylinder Σ_R of radius R. One then makes use of the boundary conditions satisfied by ϕ and G, lets $R\to\infty$, and finally integrates with respect to τ from 0 to t. Since the details of the manipulations are easily accessible [e.g., Stoker (1957), pp. 192-196, or Wehausen and Laitone (1960), pp. 604-606], we give here only the result. Define the operator $\mathcal{L}\{\phi\}$ by

$$\begin{aligned}\mathcal{L}\{\phi\}\equiv 4\pi\phi(P,t)&+\iint_S \phi(Q,t)G_\nu(P;Q;o)\,dS(Q)\\&+\int_0^t d\tau\iint_S \phi(Q,\tau)G_{\nu t}(P;Q;t-\tau)\,dS(Q).\end{aligned} \tag{12}$$

When we have need to display its variables, we shall write $\mathcal{L}\{\phi\}(P,t)$. Volterra's method leads to the following equations which must be satisfied by ϕ:

$$\begin{aligned}\mathcal{L}\{\phi\}=4\pi\phi(P,o)&+\iint_S \phi(Q,o)G_\nu(P;Q;t)\,dS\\&+\int_0^t d\tau\iint_S \dot{v}_\nu(Q,\tau)G(P;Q;t-\tau)\,dS\\&-\iint_F\left[Y_t(\xi,\zeta,o)G(P;\xi,o,\zeta;t)+YG_t\right]d\xi\,d\zeta.\end{aligned} \tag{13}$$

It is easy to verify that

$$\mathcal{L}\{\phi_t\}=\frac{\partial}{\partial t}\mathcal{L}\{\phi\}-\iint_S \phi(Q,o)\;G_{\nu t}(P;Q;t)\,dS$$

and hence that ϕ_t must satisfy

$$\begin{aligned}\mathcal{L}\{\phi_t\}=\iint_S \dot{v}_\nu(Q,t)G(P;Q;o)\,dS&+\int_0^t d\tau\iint_S \dot{v}_\nu(Q,\tau)G_t(P;Q;t-\tau)\,dS\\&-\iint_F\left[Y_t(\xi,\zeta,o)G_t(P;\xi,o,\zeta;t)+YG_{tt}\right]d\xi\,d\zeta.\end{aligned} \tag{14}$$

These, as well as equations which will appear later, can be made to yield integral equations for a function defined only on S by letting P converge to a point of the surface S. If we also call this point P, then the equations above are modified only by having 4π replaced by 2π. The solutions of these integral equations and those to appear later are unique that is, the only bounded solution of $\mathcal{L}\{\phi\}=0$ is $\phi=0$. This can be proved by a modification of Volterra's original analysis and is shown in Appendix II. We shall suppose solutions to exist for equations (13) and (14) as well as for the integral equations to appear later.

The integral equations (13) and (14) hold for ϕ_F and ϕ_W as well as for ϕ and determine each from its initial values and the boundary conditions to be satisfied on S. Hence ϕ_F satisfies exactly (13) if one replaces ϕ by ϕ_F and Y by Y_F and ϕ_W satisfies (13) with φ and Y replaced by ϕ_W and Y_W, respectively, and with the integral involving $\dot{v}_\nu$ deleted. Since $\phi_W(P,t)$ is completely determined by $\phi_W(P,o)$ and $\phi_{Wt}(x,o,z,o)$ it may seem somewhat artificial to subdivide it further as in (5). However, this corresponds to a customary way of looking at the fluid motion associated with the disturbing force. It would be possible within the framework of the present treatment to include the situation in which $\phi_I(P,t)$ is the result of specifically given pressure distributions in the free surface and/or motions of other bodies or boundaries in the absence of the body at S. By simply taking $\phi_I(P,t)$ as known, we avoid any specific assumptions about its origin.

We now make a further decomposition of ϕ_F in which

$$\phi_F(P,t) = \phi_F^{(o)}(P,t) + \phi_F^{(1)}(P,t), \tag{15}$$

where

$$\mathcal{L}\{\phi_F^{(o)}\} = 4\pi\phi_F(P,o) + \iint_S \phi_F(Q,o)G_\nu(P,Q,t)dS - \iint_F [Y_{Ft}(\xi,\zeta,o)G(P;\xi,o,\zeta;t) + Y_F G_t]\,d\xi\,d\zeta, \tag{16}$$

$$\mathcal{L}\{\phi_F^{(1)}\} = \int_o^t d\tau \iint_S \dot{v}_\nu(Q,\tau)G(P;Q;t-\tau)dS. \tag{17}$$

$\phi_F^{(o)}$ evidently describes the motion resulting from that part of the disturbance present at t=0 which has been attributed to the forced motion, and $\phi_F^{(1)}$ describes the fluid motion engendered by the body motion after t=0. Thus, even though $\phi_F^{(1)}$ satisfies

$$\phi_F^{(1)}(P,o) = 0, \quad \phi_{Ft}^{(1)}(x,o,z,o) = 0,$$

it also satisfies

$$\phi_{F_n}^{(1)}(P,t)\big|_n = v_n(P,t), \quad t > 0.$$

5. *Cummins' decomposition*

Define the function ϕ_i to be the solution of

$$\mathcal{L}\{\phi_i\} = \iint_S n_i(Q)G(P;Q;t)dS, \quad i=1,\ldots,6. \tag{18}$$

The functions ϕ_i are evidently special cases of $\phi_F^{(1)}$ where the fluid is initially at rest and one velocity or angular-velocity component undergoes a unit jump while the others remain zero. Hence ϕ_i satisfies the boundary condition

$$\phi_{in}|_S = n_i \quad \text{for} \quad t > 0. \tag{19}$$

Because of the discontinuity in v_n at t=0, we may expect that $\phi_i(P;+o)\neq 0$. This point and the equations for $\phi_i(P,+o)$ and $\phi_{it}(P,+o)$ are discussed in Appendix I. That such a motion is incompatible with the assumed linearized boundary conditions is irrelevant, for these are auxiliary functions to be used only for the purpose described below and not to describe a real motion.

We now assert that $\phi_F^{(1)}$ can be decomposed as follows:

$$\phi_F^{(1)}(P;t) = \sum_{i=1}^{6} \int_o^t \ddot{\alpha}_i(\tau)\,\phi_i(P,t-\tau)d\tau. \tag{20}$$

In fact, by direct substitution followed by a change of variables in the third integral in the definition of $\mathcal{L}$, one finds that

$$\mathcal{L}\ \{\sum_{i=1}^{6}\int_{o}^{t}\ddot{\alpha}_i(\tau)\phi_i(P,t-\tau)d\tau\}$$
$$=\sum_{i=1}^{6}\int_{o}^{t}\ddot{\alpha}_i(\tau)\mathcal{L}\{\phi_i\}(P;t-\tau)d\tau.$$

From the equation satisfied by ϕ this equals

$$\sum_{i=1}^{6}\int_{o}^{t}\ddot{\alpha}_i(\tau)d\tau\iint_{S}n_i(Q)G(P;Q;t-\tau)dS$$
$$=\int_{o}^{t}d\tau\iint_{S}\dot{v}_n(Q,t)G(P;Q;t-\tau)dS.$$

Hence (20) satisfies the integral equation (17). Since the solution of this integral equation is unique, the assertion is correct. This is essentially Cummins' decomposition (1962), as we shall see later on.

The force and moment acting upon the body which are a result of that part of the fluid motion described by $\phi_F^{(1)}$ are given in linearized theory by

$$F_i(t)=-\rho\iint_{S}\phi_{Ft}^{(1)}(P,t)n_i(P)dS(P),\quad i=1,\ldots,6, \tag{21}$$

where the force components are (F_1, F_2, F_3), the moment components are (F_4, F_5, F_6), and the n_i are as before. The decomposition above yields immediately

$$F_i(t)=-\sum_{k=1}^{6}\Big[\ddot{\alpha}_k(t)\rho\iint_{S}\phi_k(P,+o)n_i dS$$
$$+\int_{o}^{t}d\tau\,\ddot{\alpha}_k(\tau)\rho\iint_{S}\phi_{kt}(P,t-\tau)n_i dS\Big]. \tag{22}$$

Let us define

$$\mu_{ik}=\iint_{S}\phi_k(P,+o)n_i dS,\quad L_{ik}(t)=\rho\iint_{S}\phi_{kt}(P,t)n_i dS. \tag{23}$$

Then the equation for F_i may be written

$$F_i(t)=-\sum_{k=1}^{6}\Big[\mu_{ik}\ddot{\alpha}_k(t)+\int_{o}^{t}L_{ik}(t-\tau)\ddot{\alpha}_k(\tau)d\tau\Big]. \tag{24}$$

The constants μ_{ik}, which depend only upon the shape of the surface S, are the "added masses", following Cummins' use of this term.

Cummins, in making his decomposition, introduced two functions, which we shall call $\psi_k(P)$ and $\chi_k(P;t)$, instead of the one function ϕ_k. It is not difficult to show that these functions may be identified with $\phi_k(P,+o)$ and $\phi_{kt}(P,t)$, respectively. For $\phi_k(P,+o)$ it is only necessary to show that

$$\phi_{kn}|_S=n_k \quad\text{and}\quad \phi_k(x,o,z;o)=0. \tag{25}$$

The former is already satisfied and we need to show only the latter. Consider the equation satisfied by $\phi_k(P,+o)$:

$$4\pi\phi_k(P,+o)+\iint_{S}\phi_k(Q,+o)G_\nu(P;Q;o)dS$$
$$=\iint_{S}n_i(Q)G(P;Q;o)dS. \tag{26}$$

Because of the symmetry of G in P and Q it follows from the third boundary condition satisfied by G that

$$G(x, o, z; \xi, \eta, \zeta; o) = 0 \tag{27}$$

for all ξ, η, ζ. But then also

$$G_\xi(x, o, z; \xi, \eta, \zeta; o) = G_\eta = G_\zeta = 0 \tag{28}$$

and hence

$$G_\nu(x, o, z; \xi, \eta, \zeta; o) = n_1 G_\xi + n_2 G_\eta + n_3 G_\zeta = 0. \tag{29}$$

Thus the equation for $\phi_k(P, +o)$ reduces for $P=(x, o, z)$ to

$$\phi_k(x, o, z; +o) = 0.$$

Consider next $\phi_{kt}(P; t)$. It satisfies (see Appendix I) the equation

$$\mathcal{L}\{\phi_{kt}\} = \iint_S n_k(Q) G_t(P, Q; t) dS - \iint_S \phi_i(Q, +o) G_{\nu t}(P, Q; t) dS. \tag{30}$$

The boundary conditions which Cummins imposes on χ_k are the following:

$$\chi_{ktt}(x, o, z, t) + g\chi_{ky} = 0,$$

$$\chi_{kt}(x, o, z, o) + g\psi_{ky}(x, o, z) = 0,$$

$$\chi_{kn}|_S = 0,$$

$$\chi_k(P; o) = 0. \tag{31}$$

It is evident that the first two are satisfied with $\phi_{kt}(P, t)$ as χ_k and $\phi_k(P, +o)$ as ψ_k because ϕ_k satisfies the free-surface condition. The third condition follows from $\phi_{kn}|_S = n_k$ since n_k is independent of t. There remains the last one. If we set t=0 in the equation for ϕ_{kt}, it reduces to

$$4\pi\phi_{kt}(P, +o) + \iint_S \phi_{kt}(Q, +o) G_\nu(P; Q; o) dS = 0, \tag{32}$$

because $G_t(P; Q; o) = 0$. $\phi_{kt}(P, +o) = 0$ is obviously a solution. That this is the only solution is shown in Appendix II. Thus ϕ_{kt} satisfies all the conditions imposed by Cummins upon χ_k.

Cummins' formulas for $\phi_F^{(1)}$ and F_i differ from those given above in only inessential ways. Instead of starting with initial data at t=0, he starts from a state of rest at $t=-\infty$. Hence his integrals extend from $-\infty$ to t instead of from 0 to t. In addition, he has integrated once by parts. The analogous formulas in the present setting are

$$\phi_F^{(1)}(P, t) = \sum_{i=1}^{6} \Big[\dot\alpha_i(t)\phi_i(P, +o) - \dot\alpha_i(o)\phi\,(P, t) + \int_o^t \dot\alpha_i(\tau)\phi_{it}(P, t-\tau)d\tau\Big],$$

$$F_i(t) = -\sum_{k=1}^{6} \Big[\mu_{ik}\ddot\alpha_k(t) - L_{ik}(t)\dot\alpha_k(o) + \int_o^t L_{ik}(t-\tau)\dot\alpha_k(\tau)d\tau\Big]. \tag{33}$$

For a steady oscillatory motion it is well known that $F_i(t)$ can be expressed in the form

$$F_i(t) = -\sum_{i=1}^{6} \left[\mu_{ik}(\sigma)\ddot{\alpha}_k(t) + \lambda_{ik}(\sigma)\dot{\alpha}_k(t) \right], \tag{34}$$

where σ is the frequency of oscillation, and where as can be easily proved, $\mu_{ik}=\mu_{ki}$, $\lambda_{ik}=\lambda_{ki}$. The same argument shows that μ_{ik} as defined in (23) is also symmetric. This argument is not directly applicable to $L_{ik}(t)$ because of the time derivatives in the free-surface boundary condition. However, it is still possible to derive the symmetry of L_{ik} (and μ_{ik}) by a modification of this argument. Consider again the region of fluid bounded S, F, B and Σ_R. If we make use of the fact that $\phi_{kn}=0$ on B and that $\phi_k=O(R^{-2})$ as $R \to \infty$, as follows from equation (18), then from Green's Theorem applied to the region under consideration we have, with the space variables suppressed,

$$\begin{aligned} o &= \iint_S \left[\phi_k(\tau)\phi_{in}(t-\tau) - \phi_{kn}(\tau)\phi_i(t-\tau)\right] dS \\ &\quad + \iint_F \left[\phi_k(\tau)\phi_{iy}(t-\tau) - \phi_{ky}(\tau)\phi_i(t-\tau)\right] dS \\ &= \iint_S \left[\phi_k(\tau)n_i - \phi_i(t-\tau)n_k\right] dS \\ &\quad + \frac{1}{g}\iint_F \frac{\partial}{\partial\tau}\left[\phi_k(\tau)\phi_{it}(t-\tau) + \phi_{kt}(\tau)\phi_i(t-\tau)\right] dS. \end{aligned}$$

In writing the second equation use has been made of the boundary conditions on S and on the free surface. If we now integrate with respect to τ from 0 to t and recall that $\phi(x,o,z,o)=0$ and $\phi_{kt}(x,y,z,o)=0$, we find

$$o = \int_o^t d\tau \iint_S \left[\phi_k(\tau)n_i - \phi_i(t-\tau)n_k\right] dS.$$

In the second term we make the change of variables $\tau'=t-\tau$. This gives

$$\int_o^t d\tau \iint_S \phi_k(\tau)n_i dS = \int_o^t d\tau' \iint_S \phi_i(\tau')n_k dS.$$

After differentiating and multiplying by ρ, we find

$$\rho\iint_S \phi_k(t)n_i dS = \rho\iint_S \phi_i(t)n_k dS. \tag{35}$$

With t=0, this asserts that

$$\mu_{ik} = \mu_{ki}. \tag{36}$$

Taking another derivative with respect to t, we have

$$L_{ik}(t) = L_{ki}(t). \tag{37}$$

The expression for $F_i(t)$ in (34) does not include the force acting on the body as a result of the fluid motion associated with $\phi_F^{(o)}$. This can not be further decomposed or simplified. It vanishes, however, if there is no forced-motion disturbance at t=0. We shall write

$$F_i^{(o)}(t) = -\rho\iint_S \phi_{Ft}^{(o)}(P,t)n_i dS. \tag{38}$$

6. *The Haskind relations*

Let us next consider $\phi_W(P,t)$. According to (5), ϕ_W may be further decomposed into a sum of a 'known' function ϕ_I and an unknown one ϕ_D, although it is in fact determined by the integral equation for ϕ_W. The essence of the Haskind relations is that in the computations of the force and moment one may avoid solving for the diffracted wave provided one already knows the forced-wave potential. This result was established by Haskind (1957) for steady periodic motion. We derive here a similar result by applying Volterra's ideas.

We shall again use the facts that $\phi_{in}|_S = n_i$, $i=1,\dots,6$, $\phi_{in}|_B = 0$, and $\phi_i = O(R^{-2})$ as $R \to \infty$. Consider now the volume of fluid bounded by S, F, B and Σ_R, where these have the same significance as before. Since both ϕ_i and ϕ_{Dt} are harmonic in this volume, Green's Theorem gives the following:

$$\begin{aligned} o &= \iint\limits_{S+F+B+\Sigma_R} [\phi_{Dt}(P,\tau)\phi_{in}(P,t-\tau) - \phi_{Dtn}\,\phi_i]dS \\ &= \iint\limits_S \phi_{Dt}(P,\tau)n_i dS - \iint\limits_S \phi_{Dtn}(P,\tau)\phi_i(P,t-\tau)dS \\ &\quad + \iint\limits_F [\phi_{Dt}(P,\tau)\phi_{iy}(P,t-\tau) - \phi_{Dty}\,\phi_i]\,dS \\ &\quad + \iint\limits_{\Sigma_R} [\phi_{Dt}(P,\tau)\phi_{iR}(P,t-\tau) - \phi_{DtR}\,\phi_i]\,dS. \end{aligned} \tag{39}$$

As $R \to \infty$, the integral over Σ_R vanishes because of the conditions satisfied by ϕ_D and ϕ_i. Next we make use of the free-surface condition in the integral over F. This integral can be recast as follows:

$$\begin{aligned} &-\frac{1}{g}\iint\limits_F [\phi_{Dt}(P,\tau)\phi_{itt}(P,t-\tau) - \phi_{Dttt}\,\phi_i]\,d\xi\,d\zeta \\ &= \frac{1}{g}\iint\limits_F \frac{\partial}{\partial\tau}[\phi_{Dt}(P,\tau)\phi_{it}(P,t-\tau) + \phi_{Dtt}\,\phi_i]\,d\xi\,d\zeta. \end{aligned}$$

Integrate the equation (39) from 0 to t with respect to τ:

$$\begin{aligned} -\int_o^t d\tau \iint\limits_S \phi_{Dt}(P,\tau)n_i dS &= -\int_o^t d\tau \iint\limits_S \phi_{Dtn}(P,\tau)\phi_i(P,t-\tau)dS \\ &\quad + \frac{1}{g}\iint\limits_F [\phi_{Dt}(P,t)\phi_{it}(P,+o) + \phi_{Dtt}(P,t)\phi_i(P,+o)]\,d\xi\,d\zeta \\ &\quad - \frac{1}{g}\iint\limits_F [\phi_{Dt}(P,o)\phi_{it}(P,t) + \phi_{Dtt}(P,o)\phi_i(P,t)]\,d\xi\,d\zeta. \end{aligned} \tag{40}$$

Since, as shown earlier, $\phi_{it}(P,+o)=0$ for all P and $\phi_i(P,+o)=0$ for $P \in F$, the second term on the right vanishes. Having established this, we now differentiate (40) with respect to t:

$$\begin{aligned} -\iint\limits_S \phi_{Dt}(P,t)n_i dS &= -\iint\limits_S \phi_{Dtn}(P,t)\phi_i(P,+o)dS \\ &\quad - \int_o^t d\tau \iint\limits_S \phi_{Dtn}(P,\tau)\phi_{it}(P,t-\tau)dS \\ &\quad - \frac{1}{g}\iint\limits_F [\phi_{Dt}(P,o)\phi_{itt}(P,t) + \phi_{Dtt}(P,o)\phi_{it}(P,t)]\,d\xi\,d\zeta. \end{aligned} \tag{41}$$

In the integrals over S we may replace ϕ_{Dtn} by the known function $-\phi_{Itn}$. In the integral over F,

$$\phi_{Dt}(P,o)=\phi_t(P,o)-\phi_{It}(P,o),\quad \phi_{Dtt}(P,o)=\phi_{tt}(P,o)-\phi_{Itt}(P,o), \tag{42}$$

so that these may be considered as known functions.

The expression for that part of the force and moment associated with the incoming and diffracted wave may now be put in the following form:

$$\begin{aligned} F_{wi}(t) &= -\rho \iint_S \phi_{wt}(P,t)n_i(P)dS \\ &= -\rho \iint_S [\phi_{It}(P,t)\phi_{in}(P,+o)-\phi_{Itn}(P,t)\phi_i(P,+o)]\,dS \\ &\quad +\rho\int_o^t d\tau \iint_S \phi_{Itn}(P,\tau)\phi_{it}(P,t-\tau)dS \\ &\quad -\frac{\rho}{g}\iint_F [\phi_{Dt}(P,o)\phi_{itt}(P,t)+\phi_{Dtt}(P,o)\phi_{it}(P,t)]dS, \\ &\qquad\qquad i=1,\dots,6. \end{aligned} \tag{43}$$

This is the analogue of the Haskind relations.

7. *The hydrostatic restoring force and moment*

The relation of this part of the force and moment with the geometry of the body is well known. The formulas are reproduced for convenience and completeness. We introduce the following designations:

V = volume bounded by S and the waterplane,
(x_B, y_B, z_B) = center of buoyancy of V,
W = waterplane area bounded by S,
$(x_c, 0, z_c)$ = centroid of W,
WD_{xx}^2 = moment of inertia of W about (y, z)-plane,
WD_{zz}^2 = moment of inertia of W about (x, y)-plane,
WD_{xz}^2 = product of inertia of W about (x, y)- and (y, z)-planes,

$$(c_{ik}) = -\rho g \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -W & 0 & Wz_c & 0 & -Wx_c \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Wz_c & 0 & -[WD_{zz}^2+Vy_B] & 0 & WD_{xz}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -Wx_c & 0 & WD_{xz}^2 & 0 & -[WD_{xx}^2+Vy_B] \end{pmatrix} \tag{44}$$

The components of the hydrostatic restoring force and moment can now be written as follows

$$F_{HSi} = -\sum_{k=1}^{6} c_{ik}\alpha_k(t)+\rho g V\delta_{i2}-\rho g V z_B\delta_{i4}+\rho g V x_B \delta_{i6}, \tag{45}$$

where the symbol δ_{ik} has its usual significance.

8. *The equations of motion*

Let m be the mass of the body and let I_x, I_y, I_z, I_{xy}, I_{yz}, I_{zx} be its moments and products of inertia according to the usual definitions. Define the matrix (m_{ij}) as follows:

$$(m_{ij})=\begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_x & -I_{xy} & I_{xz} \\ 0 & 0 & 0 & I_{xy} & I_y & -I_{yz} \\ 0 & 0 & 0 & -I_{xz} & I_{yz} & I_z \end{pmatrix}. \tag{46}$$

The linearized equations of motion may be written in the form

$$m_{ik}\ddot{\alpha}_k(t)=F_i^{(o)}+F_i+F_{wi}+F_{HSi}, \quad i=1,\ldots,6. \tag{47}$$

The usual conditions of hydrostatic equilibrium,

$$m=\rho gV, \qquad x_B=z_B=0, \tag{48}$$

are obtained when $\alpha_k=0$ and no surface waves are present. Making use of this and introducing the forms derived earlier for F_i and F_{HSi}, we find

$$(m_{ik}+\mu_{ik})\ddot{\alpha}_k(t)+c_{ik}\alpha_k(t)+\int_o^t L_{ik}(t-\tau)\ddot{\alpha}_k(\tau)d\tau$$
$$=F_i^{(o)}(t)+F_{wi}(t), \quad i=1,\ldots,6. \tag{49}$$

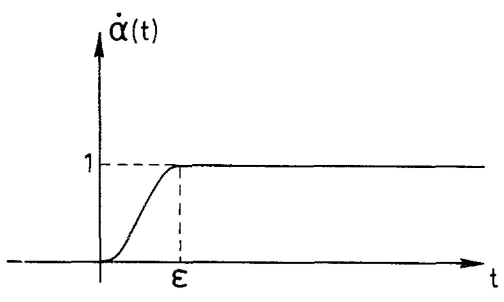
This set of integro-differential equations for $\alpha_k(t)$, with initial conditions $\alpha_k(o)$ and $\dot{\alpha}_k(o)$, together with the integral equations for $\phi_k(P;t)$, the initial motion of the fluid, $\phi(P;o)$ and $\phi_t(P;o)$, and the given incoming-wave potential $\phi_I(P,t)$, determine uniquely the behavior of both body and fluid at later instants of time. Note that the coefficients μ_{ik}, c_{ik} and the kernel L_{ik} depend only upon geometrical properties of the body and not upon the motion itself. This is, of course, the great virtue of Cummins' decomposition. No attempt at further analysis of (49) will be made here.

APPENDIX I

Consider a motion for which

$$v_n(P,t)=\dot{\alpha}(t)n_i \tag{A.1}$$

for some i, $1\leqq i\leqq 6$, and let $\dot{\alpha}(t)$ be a function of the form shown below.



The associated velocity potential $\phi_F^{(1)}$ will be denoted by $\phi_i^{(\epsilon)}$. It satisfies the integral equation

$$\mathcal{L}\{\phi_i^{(\epsilon)}\}=\int_o^t d\tau\, \ddot{\alpha}(\tau)\iint_S n_i(Q)G(P;Q;t-\tau)dS.$$

For $t \geqq \epsilon$, the right-hand side becomes

$$= \int_0^\epsilon d\tau\, \ddot{\alpha}(\tau) \iint_S n_i(Q)G(P;Q;t-\tau)dS$$
$$= \iint_S n_i(Q)G(P;Q;t-\epsilon)dS$$
$$+ \int_0^\epsilon d\tau\, \dot{\alpha}(\tau) \iint_S n_i(Q)G_t(P;Q;t-\tau)dS.$$

As $\epsilon \to 0$, the integral equation approaches

$$\mathcal{L}\{\phi_i\}(P,t) = \iint_S n_i(Q)G(P;Q;t)dS. \qquad (A.2)$$

The integral equation for ϕ_{it} must be found directly from (A.2) because of the discontinuous behavior at $t=0$.
It is

$$\mathcal{L}\{\phi_{it}\} = \iint_S n_i(Q)G_t(P;Q;t)dS - \iint_S \phi_i(Q,o)G_{\nu t}(P;Q;t)dS. \qquad (A.3)$$

If one now lets $t \to +o$, the equations for $\phi_i(P,+o)$ and $\phi_{it}(P,+o)$ become, respectively,

$$4\pi\phi_i(P,+o) + \iint_S \phi_i(Q,+o)G_\nu(P;Q;o)dS$$
$$= \iint_S n_i(Q)G(P;Q;o)dS, \qquad (A.4)$$

$$4\pi\phi_{it}(P,+o) + \iint_S \phi_{it}(Q,+o)G_\nu(P;Q;o)dS = 0. \qquad (A.5)$$

It is shown in Appendix II that the solutions are unique. In particular, it follows that $\phi_{it}(P,+o)=0$.

APPENDIX II

In order to see that

$$\phi_{kt}(P;+o) = 0 \qquad (A.6)$$

is the unique solution of the equation (A.5) for $\phi_{kt}(P;+o)$, consider any arbitrary solution and define

$$U(P) = \iint_S \phi_{kt}(Q;+o)G_\nu(P;Q;o)dS. \qquad (A.7)$$

The function U is defined both inside and outside S and is harmonic. It follows from equation (26) above that

$$U(x,o,z) = 0. \qquad (A.8)$$

Let P converge to a point of S from the inside. Then from a well-known property of such integrals

$$\lim_{P_{int}\to P\in S} U(P_{int}) = 2\pi\phi_{kt}(P,+o) + \iint_S \phi_{kt}(Q,o)G_\nu(P;Q;o)dS. \qquad (A.9)$$

Next, in the equation (A.5) satisfied by $\phi_{kt}(P,+o)$ for P exterior to S let P converge to a point of S from the outside. This yields

$$2\pi\phi_{kt}(P,+o) + \iint_S \phi_{kt}(Q,+o)G_\nu(P;Q;o)dS = 0. \qquad (A.10)$$

Hence

$$\lim_{P_{int}\to P\in S} U(P_{int}) \equiv U_{int}(P) = 0. \qquad (A.11)$$

It now follows that $U\equiv 0$ in the interior region bounded by S and F (or by S alone if the body is completely submerged); one may, for example, derive this from Green's Theorem:

$$\iiint_V (\text{grad } U)^2\, dV = \iint_{S+F} UU_n\, dS. \qquad (A.12)$$

Since $U\equiv 0$ inside S, then also $U_n|_S = 0$ inside S. But then, since $\phi_{kt}(P,+o)$ is continuous, U_n is continuous across S [see, e.g., Kellogg (1929), p. 170, Th. X]. Hence $U_n|_S = 0$ outside S. Since $G_{\nu|Q\in B} = 0$, then also $G_{n|P\in B} = 0$ for all Q. From this follows

$$G_{n\xi} = G_{n\eta} = G_{n\zeta} = 0 \quad \text{for} \quad P\in B \qquad (A.13)$$

and hence

$$G_{n\nu} = 0 \quad \text{for} \quad P\in B,\ Q\in S. \qquad (A.14)$$

We thus deduce that

$$U_{n|B} = 0. \qquad (A.15)$$

Further,

$$U = O(R^{-2}) \quad \text{if} \quad R\to\infty \qquad (A.16)$$

from the assumed behavior of G. We may now once again apply the Green Theorem above to conclude that $U\equiv 0$ outside S, i.e., in the region of fluid. But now it follows from the equation for $\phi_{kt}(P,+o)$ that $\phi_{kt}(P,+o)\equiv 0$, as we wished to show. It follows immediately that the solution to (A.4) is unique since the difference of two solutions would satisfy (A.5).

Next we wish to show that

$$\mathcal{L}\{\phi\}(P,t) = 0 \qquad (A.17)$$

implies that

$$\phi(P,t) = 0. \qquad (A.18)$$

Define now

$$U(P,t) = \iint_S \phi(Q,t) G_\nu(P;Q;o)\,dS$$

$$+ \int_o^t d\tau \iint_S \phi(Q,\tau) G_{\nu t}(P;Q;t-\tau)\,dS. \qquad (A.19)$$

The proof above shows that $U(P,o)=0$ if $\mathcal{L}\{\phi\}(P,o)=0$.

Since $U(P,t)$ is defined inside S, we may, as above, find for a point $P \epsilon S$

$$\lim_{P_{int} \to P} U(P_{int}, t) = 2\pi\phi(P,t) + U(P,t). \qquad (A.20)$$

On the other hand, from $\mathcal{L}\{\phi\}(P,t)=0$ follows, by taking the limit from the exterior,

$$2\pi\phi(P,t) + U(P,t) = 0. \qquad (A.21)$$

Hence

$$\lim_{P_{int} \to P} U(P_{int}, t) = U_{int}(P,t) = 0, \quad P \epsilon S, \qquad (A.22)$$

and then also

$$\frac{\partial}{\partial t} U_{int}(P,t) = 0, \quad P \epsilon S. \qquad (A.23)$$

Consider now the boundary $y=0$. It follows from (5a) and (6) that

$$G_{tt}(x,o,z;Q;t) + gG_y \equiv DG(x,o,z;Q;t) = 0 \qquad (A.24)$$

and thence that

$$DG_\nu = DG_{\nu t} = 0 \quad \text{on} \quad y=0. \qquad (A.25)$$

We wish to show that also

$$DU = 0 \quad \text{on} \quad y=0. \qquad (A.26)$$

One finds, using $G_{\nu t}(x,o,z;Q,o) = G_\nu(x,o,z;Q;o) = 0$,

$$U_{tt} = \iint_S \phi(Q,t) G_{\nu tt}(P;Q;o)\,dS$$

$$+ \int_o^t d\tau \iint_S \phi(Q,\tau) G_{\nu ttt}(P;Q;t-\tau)\,dS$$

and hence

$$DU(P,t)\Big|_{y=0} = \iint_S \phi(Q,t)\left[G_{\nu tt}(P;Q;o) + gG_{\nu y}\right]_{y=0} dS$$

$$+ \int_o^t d\tau \iint_S \phi(Q,\tau)\left[G_{\nu ttt}(P;Q;t-\tau) + gG_{\nu ty}\right]_{y=0} dS,$$

which is zero from (A.25).

Consider next the volume V bounded by S and F and define

$$E_{int}(t) = \tfrac{1}{2} \iiint\limits_V |\nabla U|^2 \, dV + \frac{1}{2g} \iint\limits_F U_t^2 \, dS. \tag{A.27}$$

Then, where $\underline{n}$ is the normal pointing into the region,

$$\begin{aligned} E'_{int}(t) &= \iiint\limits_V \nabla U \cdot \nabla U_t \, dV + \frac{1}{g} \iint\limits_F U_t U_{tt} \, dS \\ &= - \iint\limits_S U_t U_n \, dS + \iint\limits_F U_t \left[U_y + \frac{1}{g} U_{tt} \right] dS. \end{aligned}$$

The first integral vanishes because of (A.23), the second because of (A.26). Hence

$$E_{int}(t) = E_{int}(o) = 0, \tag{A.28}$$

because $U(P,o)=0$. Now it follows from (A.27) and (A.28) that

$$\nabla U = U_t = 0.$$

Since $U(P,o)=0$, then

$$U(P,t) = 0 \quad \text{inside } S. \tag{A.29}$$

Just as for $t=0$, we may conclude from (A.28) that $U_n|_S=0$ on the inside of S and hence also on the outside.

Consider next the volume bounded by S, F, B and Σ_R. The reasoning leading to (A.15) and (A.16) holds also for $U(P,t)$. Define

$$E_{ext}(t) = \tfrac{1}{2} \iiint\limits_V (\nabla U)^2 + \frac{1}{2g} \iint\limits_F U_t^2 \, dS. \tag{A.30}$$

Then,

$$\begin{aligned} E'_{ext}(t) &= \iint\limits_S U_t U_n \, dS + \iint\limits_F U_t \left(U_y + \frac{1}{g} U_{tt} \right) dS \\ &\quad + \iint\limits_{\Sigma_R} U U_R \, dS. \end{aligned}$$

The integral over S vanishes because $U_n=0$, that over F because of (A.26), and the last one vanishes as $R \to \infty$. Hence,

$$E_{ext}(t) = E_{ext}(o) = 0. \tag{A.31}$$

As above, we conclude again that

$$U(P,t) = 0 \quad \text{outside } S. \tag{A.32}$$

It now follows immediately from

$$\mathcal{L}\{\phi\}(P,t) = 4\pi\phi(P,t) + U(P,t) = 0$$

that

$$\phi(P,t) = 0. \tag{A.33}$$

APPENDIX III

It has been mentioned earlier that uniqueness of solution of the various integral equations follows from a slight modification of Volterra's original argument. The question of existence of solution can be reduced to the same question for a Fredholm integral equation of the second kind.

In order to see this, we introduce the following Laplace transforms:

$$\tilde{\phi}(P, s) = s \int_0^\infty e^{-st} \phi(P, t)dt$$

$$= \phi(P, +o) + \int_0^\infty e^{-st} \phi_t(P, t)dt$$

$$= \phi(P, +o) + \frac{1}{s} \phi_t(P, +o) + \frac{1}{s}\int_0^\infty e^{-st} \phi_{tt}(P, t)dt, \qquad (A.34)$$

$$\tilde{v}_\nu(P, s) = s \int_0^\infty e^{-st} v_\nu(P, t)dt, \qquad (A.35)$$

$$\tilde{G}(P; Q; s) = s \int_0^\infty e^{-st} G(P; Q; t)dt. \qquad (A.36)$$

It is straightforward to confirm that the integral equations for ϕ and ϕ_k take the following forms after transformation:

$$4\pi\tilde{\phi}(P, s) + \iint_S \tilde{\phi}(Q, s)\tilde{G}_\nu(P; Q; s)dS$$

$$= 4\pi\phi(P, o) + \iint_S \phi(Q, o)\tilde{G}_\nu(P; Q; s)ds$$

$$+ \iint_S [\tilde{v}_\nu(Q, s) - v_\nu(Q, o)]\tilde{G}(P; Q; s)ds$$

$$- \iint_F [Y_t(\xi, \zeta, o) + sY(\xi, \zeta, o)]\tilde{G}(P; \xi, o, \zeta; s)d\xi\, d\zeta, \qquad (A.37)$$

$$4\pi\tilde{\phi}_k(P, s) + \iint_S \tilde{\phi}_k(Q, s)\tilde{G}_\nu(P; Q; s)dS = \iint_S n_i(Q)G(P; Q; s)dS. \qquad (A.38)$$

The integral equations for the Laplace transforms of the other potential functions which have been introduced can be deduced immediately from the first one above.

The function $\tilde{G}(P; Q; s)$ has the same structure as $G(P; Q; t)$, i.e.,

$$\tilde{G}(P; Q; s) = \frac{1}{r} + \tilde{H}(P; Q; s) \qquad (A.39)$$

where $\tilde{H}$ is harmonic in the region $y < 0$. Hence, if one lets P converge to a point of the surface S, the 4π's above are replaced by 2π's and the equations appear to be typical of those occurring in Neumann problems in potential theory. This is not the case, however, for the surface S is not necessarily closed and the condition satisfied by $\tilde{G}$ on the free surface,

$$s^2\tilde{G}(x, y, z; \xi, o, \zeta; s) + g\tilde{G}_\eta = 0 \qquad (A.40)$$

does not allow reflection of the lower half-space into the upper one. Hence, one must deal directly with the integral equations (A. 37) or (A. 38). In the case of steady harmonic oscillation, in which the sign of the first term in (A.40) is changed, Fritz John (1951), has been able to prove existence of a solution for a similar equation under certain conditions, so that it seems reasonable to be hopeful in the present case.

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